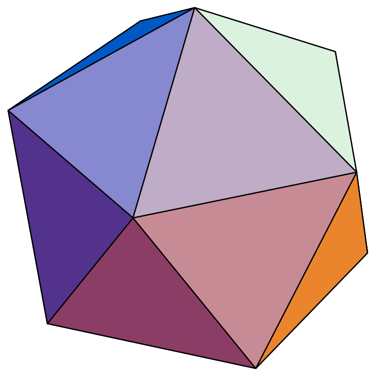
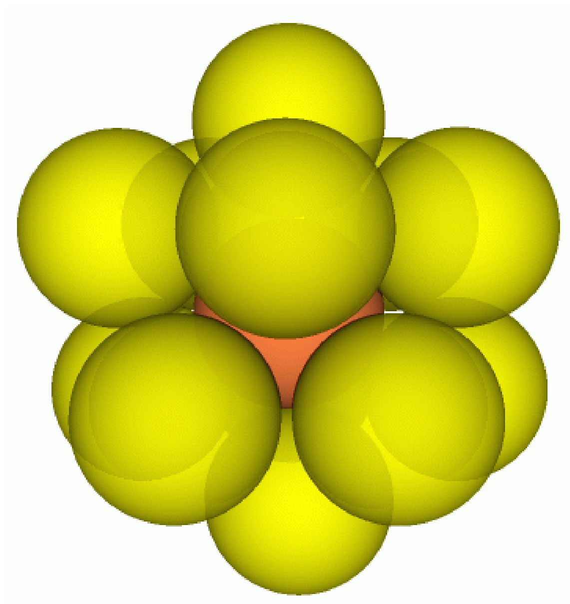
The Problem of the Thirteen Spheres

One can easily arrange 12 unit spheres all touching a central one:



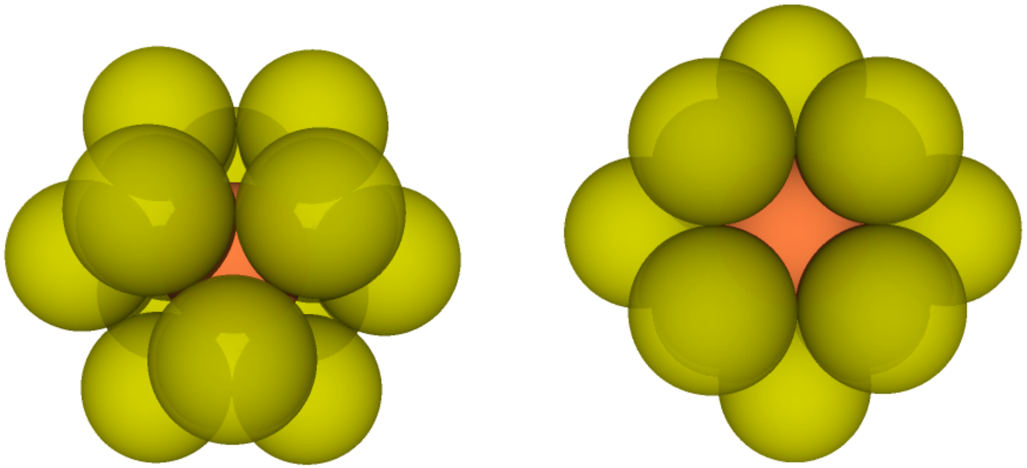


For example, touching the central one at the 12 vertices of an inscribing regular icosahedron.

Note: This arrangement is very untight.

$$4 \sin \frac{\tan^{-1} 2}{2} = 2.102924\ldots$$

In fact, there is another arrangement of 12 touching neighbors, called the f.c.c. configuration:



There are six "*big holes*" in this configuration, as indicated in the figure.

The Problem of the 13 spheres:

"*Is it possible to create a hole big enough to allow an additional 13th touching neighbor?*"

There was a recorded discussion between David Gregory and Isaac Newton in 1694. It was believed that they had the following viewpoints:

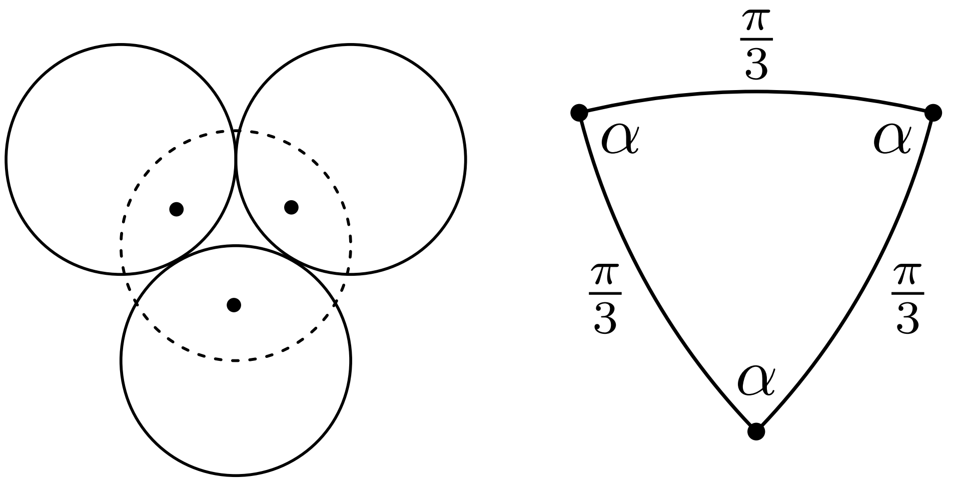
Newton: "12 should be the maximal."

Gregory: "13 might be possible."

Also known as Newton's Problem.

...turns out to be a challenging problem.

Intuitively, tightest local arrangement of 3 touching neighbors should look like:



The three touching points on the central sphere will form a $\frac{\pi}{3}$-equilateral spherical triangle with area $\triangle_{\frac{\pi}{3}}$, where:

$$\triangle_{\frac{\pi}{3}} = 3\alpha - \pi, \quad \alpha = \cos^{-1}\frac{1}{3}.$$

Euler formula: $v - e + f = 2$. For triangulations, $3f = 2e$.

The sphere will be subdivided into $f = 2v - 4$ triangles.

Direct calculations:

$$\begin{aligned} 12 \text{ pts} : &\quad 4\pi - 20\triangle_{\frac{\pi}{3}} = 1.5406\ldots \\ 13 \text{ pts} : &\quad 4\pi - 22\triangle_{\frac{\pi}{3}} = 0.4380\ldots \\ 14 \text{ pts} : &\quad 4\pi - 24\triangle_{\frac{\pi}{3}} = -0.6644\ldots \end{aligned}$$

So, in terms of total area accounting (with certain separation requirement),

"13 touching neighbors *might be possible*".

Answer: 13 is *impossible*.

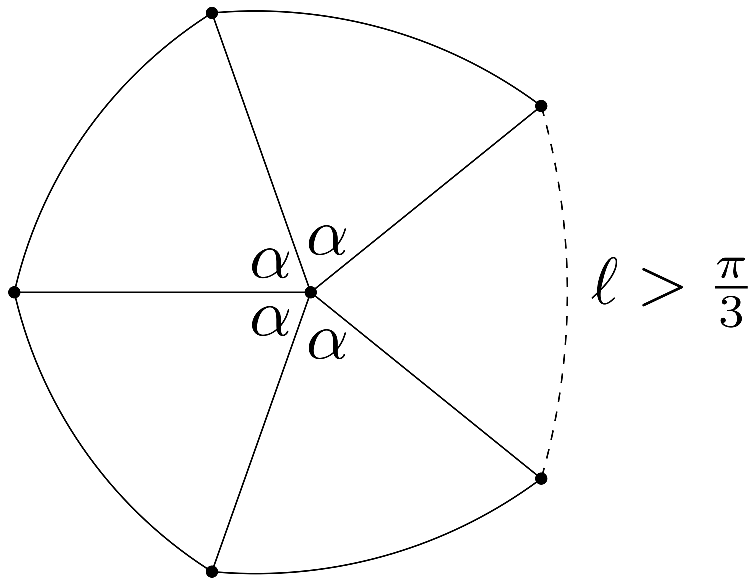
1694	recorded discussion
1874-5	some (incorrect) proofs
1953	first (2) correct proofs by Schütte & van der Waerden
1956	another proof sketched by Leech

Recently,

1993	W.Y. Hsiang
1998	M. Aigner & G. Ziegler
2003	K. Böröczky
2004	K. Anstreicher
2006	O. Musin
2007	H. Maehara

Roughly speaking, because of the inequality

$$5\alpha < 2\pi < 6\alpha, \quad \alpha = \cos^{-1}\frac{1}{3}$$



it is impossible to have a tight local arrangement.

i.e. need to use up some additional area when piecing the triangles together.

The proof by van der Waerden:

1. construction of irreducible graph with edges of **equal** lengths.

2. local estimation on "angle-excesses" of a polygon (or a collection of polygons around a vertex).

3. estimates in (2.) contradict with global estimation on angle-excesses.

- the construction of "irreducible graph" is non-trivial.

- required to perform deformations on a "hypothetical" configuration.

The proof "sketched" by Leech:

1. construction of a graph just by specific choices on edge-length bounds.

2. local estimation on area-excess for individual polygons.

3. possible combinatorial types satisfying estimates in (2.) and total area-excess estimate actually can **never** exist.

- lower bound estimate in (2.) turns out to be non-trivial.

- Leech: "*certain details which are tedious rather than difficult being omitted*".

- Leech: "*I know of no better proof of this than sheer trial*".

The proof by Hsiang:

1. graph obtained by radial projection of the Euclidean convex hull of the vertices.
2. lower bound area estimations of a collection of polygons around a vertex.
3. 13 vertices $\Rightarrow$ the existence of vertex with degree ≥ 6.
4. the area-excess of a $\frac{\pi}{3}$-saturated
   $\begin{cases} 6\triangle\text{-star} \\ 7\triangle\text{-star} \end{cases} >$ total area-excess,
   contradiction.

- the lower bound estimate in (2.) is highly non-trivial.

A qualitative comparison:

proof by	the graph constructed	area estimates	combinatorial analysis
SW	sophisicated	simple	simple
Leech	simple, artificial	a bit involved	a bit involved
Hsiang	simple, natural	rather involved	trivial

Upper bound estimations on δ_{13}:

δ_{13}: maximal spherical separation for placing 13 points on the unit sphere. ($\frac{\pi}{3} = 1.04719\ldots$)

SW	1.04318
Leech	1.04635
Hsiang	1.04455 (1.02746)

Conjecture: $\delta_{13} = 0.99722359\ldots$

claimed to be Yes by O. Musin and A. Tarasov, 2015 arXiv involves computer elimination of almost 100 million graphs.

Spherical Geometry (on unit sphere):

Lemma 1: (Area formula)

$$\triangle = \angle A + \angle B + \angle C - \pi,$$

$$\text{or } \tan\frac{\triangle}{2} = \frac{D}{u}$$

where $D = \det(\mathbf{a}, \mathbf{b}, \mathbf{c}) > 0$, $u = 1 + \cos a + \cos b + \cos c$.

By product formula of determinant, we have:

$$D^2 = 1 + 2\cos a \cos b \cos c - \cos^2 a - \cos^2 b - \cos^2 c.$$

Lemma 2: Let **a**, **b**, **c**, **d** be the vertices of a quadrilateral, and let $\overrightarrow{OV}_1$ and $\overrightarrow{OV}_2$ be given by:

$$\overrightarrow{OV_1} = \frac{1}{\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}} \{\mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}\},$$
$$\overrightarrow{OV_2} = \frac{1}{\mathbf{a} \times \mathbf{c} \cdot \mathbf{d}} \{\mathbf{a} \times \mathbf{c} + \mathbf{c} \times \mathbf{d} + \mathbf{d} \times \mathbf{a}\}.$$

Then:

$$\overrightarrow{V_1 V_2} = \frac{d\Box}{dt} \frac{\mathbf{a} \times \mathbf{c}}{|\mathbf{a} \times \mathbf{c}|}, \quad \frac{d\Box}{dB} = \overrightarrow{V_2 V_1} \cdot \mathbf{b}.$$

Corollary: A quadrilateral with four given side-lengths attains its maximal area when it is cocircular. Shearing deformation further away from cocircularity is monotonic area-decreasing.

Lemma 3 (Lexell's Theorem): Let $\triangle ABC$ and $\triangle ABC'$ have the same oriented area. Then C, C', antipodal points of A and B are cocircular.

Corollary: Cluster of isosceles triangles with a fixed sum of central angles, more lopsided distribution $\Rightarrow$ smaller total area.